Moduli Space for Solutions of Non-Abelian Vortices

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§1. Introduction & Motivation

- Vortex is a magnetic flux (with two codim.) squeezed in Higgs phase and is a stable soliton as topological defects supported by winding $U(1)$-broken vacuum; $\pi_1(U(1)) = \mathbb{Z}$ which is counted by

$$k = -\frac{1}{2\pi}\int dx^1 dx^2 F_{12} \in \mathbb{Z}.$$

- In the first place, system for vortex (with $U(1)$ gauge symmetry) and their numerical configuration was proposed by Abrikosov and Nielsen-Olesen (ANO).
- Proof of existence and uniqueness for multi ANO vortex and their moduli space was given by Taubes and scattering of two vortices were considered by Manton and Speight.
- Ann Davis discussed vortices (cosmic strings) as 1/2 BPS states in the $N = 1, d = 4$ supersymmetric theory.

• Extension to vortices in non-Abelian gauge theory was discussed by Hanany-Tong, Konishi et al.

• Moduli spaces for 1/2 BPS states in non-Abelian gauge theory were determined by

	codim.	BPS solutions	D-brane construction
instantons	4	ADHM ('78)	Witten, Douglas('95)
monopoles	3	Nahm ('80)	Green, et al., Diaconescu ('96)
vortices	2	EINOS('05?)	Hanany-Tong ('03)
(domain-)walls	1	INOS ('04)	EINOO'S ('04)

Investigation of the moduli space for non-Abelian vortices can be applied to ...

• Reconnection of cosmic strings

Polchinski,..., Hashimoto-Hanany, Hashimoto-Tong

• brane world scenario

An effective theory on 1/2 BPS vortices in $d = 6$ is $d = 4, \mathcal{N} = 1$ SUSY theory, and moduli parameters are promoted to massless particles.

• 1/4 BPS composite states of vortices and other solitons

§2. 1/2 BPS Equations for Vortices

- Our model: $d=4, \mathcal{N}=2$ SUSY $U(N)$ gauge theory
  with N fundamental hypermultiplets
  (with 8 supercharges $(d,\mathcal{N})=(6,1),(5,1),(4,2),(3,4))$

Field contents (bosonic part): $(\mu,\nu=0,1,2,3, \quad \alpha=1,2)$

Vector multiplet: gauge field W_μ, adjoint scalar Σ_α,

Hyper multiplets: complex $N\times N$ matrix $(H^i)^{rA}\equiv H^{irA}$,
$SU(2)_R$ $i=1,2$, color $r=1,\cdots,N$, flavor $A=1,2,\cdots N$

Our Lagrangian (bosonic part)

$$\mathcal{L}\Big|_{\text{bosonic}} = -\frac{1}{2g^2}\mathrm{Tr}[(F_{\mu\nu}(W))^2]+\frac{1}{g^2}\mathrm{Tr}[(\mathcal{D}_\mu\Sigma)^2] + (\mathcal{D}_\mu H)^\dagger_{iAr}\mathcal{D}^\mu H^{irA} - V_{\text{pot}}$$

The scalar potential of this model

$$V_{\rm pot} = \frac{g^2}{4}{\rm Tr}\left[\left(c^a - (\sigma^a)^j{}_i H^i H^\dagger_j\right)^2\right] + H^\dagger_{iAr}[\Sigma^2]^r{}_s H^{isA}$$

Fayet-Illiopoulos term: $c_a = (0, 0, c > 0)$

Vacuum is given by

$$H^1 = \sqrt{c}\mathbf{1}_N, \quad H^2 = 0, \quad \Sigma_\alpha = 0$$

which is so called color-flavor locking vacuum with a breaking pattern

$$\mathcal{G} = U(N)_{\rm G} \otimes SU(N)_{\rm F} \quad \longrightarrow \quad \mathcal{H} = SU(N)_{\rm G+F}$$

ensures existence of vortex solution

$$\pi_1(\mathcal{G}/\mathcal{H}) = \pi_1(U(N)) = \mathbb{Z}.$$

•1/2 BPS equations for vortices

By requiring that the unbroken SUSY transformations of fermions vanishes, we find a set of BPS equations

$$\begin{aligned} 0 &= \mathcal{D}_1 H^1 + i\mathcal{D}_2 H^1, \\ 0 &= F_{12}(W) + \frac{g^2}{2}(c - H^1 H^{1\dagger}) \end{aligned}$$

$$(\Sigma_\alpha = H^2 = 0)$$

and solutions depend on only a coordinate x^1, x^2.

- Bogomol'nyi bound for vortices

$$\begin{aligned} \mathcal{E} &= \int dx^1 dx^2 (\text{r.h.s of BPS eqs.})^2 + T_{\text{vortices}} \\ &\geq T_{\text{vortices}} = -c \int dz d\bar{z} \mathrm{Tr} F_{12}(W) = 2\pi c k, \quad k \in \mathbb{N} \end{aligned}$$

Tension for 1/2 BPS vortices is given by the topological charges.

§3. 1/2 BPS solutions for vortices and their moduli space

with complex coordinate $z = x^1 + ix^2$, $\bar{\partial} = \partial/\partial z^*$

$$H^1(z, z^*) = S^{-1}(z, z^*) H_0(z), \quad \bar{\partial} H_0(z) = 0$$
$$W_1 + iW_2 = -i2S^{-1}\bar{\partial}S$$

with an arbitrary $N \times N$ matrix $H_0(z)$, and an $S(z, z^*) \in GL(N, C)$.

The second BPS eq. can be rewritten to
'Master equation' for a gauge invariant quantity $\Omega \equiv SS^\dagger$

$$\partial\bar{\partial}\Omega - (\bar{\partial}\Omega)\Omega^{-1}(\partial\Omega) = \frac{g^2}{4}\left(c\Omega - H_0(z)H_0^\dagger(z^*)\right)$$

cf. proof of existence and uniqueness in Abelian case: Taubes

Physical fields $W_{1,2}, H^1$ can be obtained by given H_0,

$$H_0 \to \Omega \to S \to W_{1,2},\ H^1$$

H_0 parametrize the moduli space for vortices.
We call H_0 a 'moduli matrix'.

An equivalence relation

$$H_0(z) \simeq V(z)H_0(z), \quad V(z) \in \mathrm{GL}(N,\mathrm{C}),\ \bar{\partial}V(z) = 0$$

Boundary condition for k vortices

$$T_{\text{vortices}} = 2\pi c k = \frac{c}{2}\oint dz\partial \log(\det H_0) + c.c.$$

$$\leftrightarrow \qquad \det(H_0) \sim z^k \quad \text{for} \ \ z \gg 1$$

$$(\det(H_0(z_i)) = 0, \quad \rightarrow \quad z_i\text{: position of a vortiex})$$

and each point on the boundary S^1 $\in$ a gauge equivalence class

$\Rightarrow H_0 \in \mathrm{Pol}(z)$: polynomial functions of z

Moduli Space of k vortices,

$$\mathcal{M}_{N,k} = \frac{\{H_0(z)|H_0(z) \in N \times N \text{ matrix},\ H_0(z) \in \mathrm{Pol}(z),\ \det(H_0) \sim z^k\}}{\{V(z)|V(z) \in \mathrm{GL}(N,\mathrm{C}),\ \bar{\partial}V(z) = 0\}}$$

- coordinates for patches of the moduli space

There are several patches for the moduli manifold. Patches are given by the following form, #patches= $\binom{k+N-1}{k}$

$$H_0(z) = \begin{pmatrix} z^{k_1} & 0 & \cdots & 0 \\ 0 & z^{k_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & z^{k_N} \end{pmatrix} - \begin{pmatrix} T^1{}_1(z) & T^1{}_2(z) & \cdots & T^1{}_N(z) \\ T^2{}_1(z) & T^2{}_2(z) & \cdots & T^2{}_N(z) \\ \vdots & & \ddots & \vdots \\ T^N{}_1(z) & \cdots & & T^N{}_N(z) \end{pmatrix},$$

where $T^r{}_s(z)$ are polynomial functions with degree less than $k_s - 1$.
We find $\dim(\mathcal{M}_{N,k}) = 2kN$. cf. Hanany-Tong

A transition function of coordinates between two patches, H_0 and H_0' is given by the V-equivalence relation, $H_0'(z) = V(z)H_0(z)$.

$\Downarrow$

In principle, the moduli manifold for k vortices can be uniquely determined.

- Example for single vortex

N patches of moduli manifold for a vortex

with $z_0, b_r, (b'_r) \in \mathbb{C}$

$$H_0 \simeq \begin{pmatrix} 1 & & 0 & -b_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -b_{N-1} \\ 0 & \dots & 0 & z - z_0 \end{pmatrix} \simeq \begin{pmatrix} 1 & & -b'_1 & 0 \\ & \ddots & \vdots & \\ 0 & & z - z_0 & 0 \\ 0 & \dots & -b'_N & 1 \end{pmatrix} \simeq \cdots$$

The V-equivalence relation gives relations between N patches

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{N-1} \\ 1 \end{pmatrix} = b_{N-1} \begin{pmatrix} b'_1 \\ \vdots \\ 1 \\ b'_N \end{pmatrix} = \cdots = b_1 \begin{pmatrix} 1 \\ \vdots \\ b''_{N-1} \\ b''_N \end{pmatrix}$$

$\{b_r\} \in \mathbb{C}P^{N-1} \simeq SU(N)/(U(1) \otimes SU(N-1))$: orientation of a vortex.

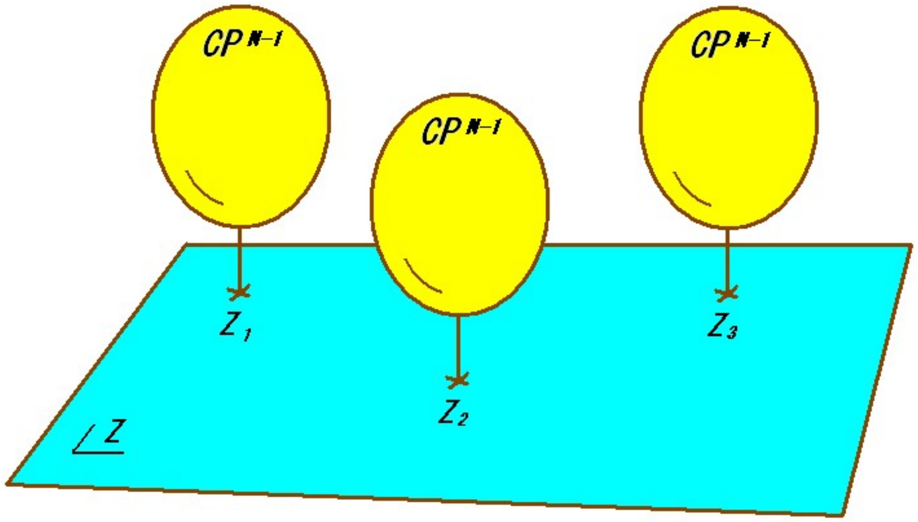
the moduli space : $\mathcal{M}_{N,k=1} = \mathbb{C} \times \mathbb{C}P^{N-1}$

cf. Hanany-Tong, Konishi et al.

• general property of moduli matrix for k vortices

We find that the moduli space of separated k vortices in $U(N)$ gauge theory is given by the following.

$$\mathcal{M}_{N,k} \sim \frac{\left(\mathbf{C} \times \mathbf{C}P^{N-1}\right)^k}{S_k}$$



C^k : positions of k vortices

$\mathbf{C}P^{N-1}$: orientation of each vortex

S_k : permutation group

cf. Abelian case: Taubes
non-Abelian case: Hanany-Tong, Hashimoto-Tong

If we consider coincident vortices, we need other analysis.

- Cases with $N_{\rm F} > N_{\rm C}$

Dimension of the moduli space turns out to be,

$$\dim(\mathcal{M}_{N_{\rm F},N_{\rm C},k}) = 2kN_{\rm F}.$$

Their patches of coordinates and their transition between patches also can be determined.

- Effective action on vortices

If we obtain an exact solution of $\Omega \equiv e^{2V}$ with given H_0, we can obtain a Kähler potential by

$$K(\phi,\phi^*) = \int d^2z \mathrm{Tr}\left[2cV + e^{-2V}H_0H_0^\dagger + \frac{16}{g^2}\int_0^1 dx \int_0^x dy \bar\partial V e^{2yL_V}\partial V\right]$$

with $L_V X \equiv [V, X]$. cf. Abelian case; Samols, Chen-Manton,

In the case with semi-local vortices ($N_{\rm F} > N_{\rm C}$) and strong coupling limit, we obtain

$$K(\phi,\phi^*) = c\int d^2z \log\det\left(H_0(z,\phi)H_0^\dagger(z^*,\phi^*)\right).$$

§4. Summary and Discussion

- We determined moduli spaces for 1/2 BPS non-Abelian vortices from the point of view of field theory.

- We partially relate our result to the result given by Hanany-Tong: the Higgs branch on an $U(k)$ gauge theory with

an adjoint chiral multiplet Z and N fundamental chiral multiplets Ψ.

There are many future problem.

- We need a complete relation between $H_0(z) \leftrightarrow Z, \Psi$

- proof of existence and uniqueness for the master equations of Ω

- method for approximation of solutions

- Generalization: non-minimal kinetic term, SUGRA, adjoint scalars, other gauge group,. . .

§5. Appendix

By using of V-equivalence relation, H_0 is always rewritten to one of the following representatives of H_0 without any redundancy, which correspond to the moduli space of vortices by one-to-one.

$$H_0 = \begin{pmatrix} P_1(z) & R_{2,1}(z) & R_{3,1}(z) & \cdots & R_{N,1}(z) \\ 0 & P_2(z) & R_{3,2}(z) & \cdots & R_{N,2}(z) \\ \vdots & & \ddots & & \vdots \\ & & & P_{N-1}(z) & R_{N,N-1}(z) \\ 0 & \cdots & & 0 & P_N(z) \end{pmatrix}$$

$$\text{with } P_r(z) = \prod_{i=1}^{k_r}(z - z_{r,i}), \quad R_{r,m}(z) \in \mathrm{Pol}(z;k_r).$$

where $\mathrm{Pol}(z;n)$ is a set of polynomial fun. of z with degree less than n.

Boundary condition requires

$$\det(H_0) = \prod_{r=1}^{N} P_r(z) \sim z^k, \qquad \leftrightarrow \qquad \sum_{r=1}^{N} k_r = k$$

$\Rightarrow \dim(\mathcal{M}_{N,k}) = 2kN \geq \sum_{r=1}^{N} 2k_r r$ cf. Hanany-Tong

(Equality is given by a case with $k_r = 0, (r \neq N), k_N = k$.)

- general property of moduli matrix for k vortices

Let us consider a typical patch

$$H_0 \simeq \begin{pmatrix} 1 & & 0 & -R_1(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -R_{N-1}(z) \\ 0 & \dots & 0 & P(z) \end{pmatrix}$$

with $P(z) \equiv \det(H_0) = \prod_{i=1}^{k}(z - z_i)$
and $(\vec{R}(z))^r = R_r(z) \in \mathrm{Pol}(z; k), r = 1, \dots N-1.$

Flavor transformation $SU(N)$ and its pull-back,

$$\delta H_0(z) = v(\xi, z) H_0(z) + H_0(z) u(\xi), \qquad \text{with } u(\xi) = \begin{pmatrix} 0 & -\vec{\xi} \\ \vec{\xi}^\dagger & 0 \end{pmatrix},$$

work on moduli parameters (functions) as,

$$\begin{aligned} \delta\vec{R}(z) &= \vec{\xi} + \vec{R}(z)(\xi^\dagger \cdot R(z)) + \vec{s}_{\xi^\dagger}(z) P(z) \in \mathrm{Pol}(z; k) \\ &\equiv \vec{\xi} + \vec{R}(z)(\xi^\dagger \cdot R(z)) \quad \mathrm{mod}\ P(z). \end{aligned}$$

with $\vec{s}_{\xi^\dagger}(z) \in$ Pol(z).
Here 'modulo $P(z)$' defines a map from $\mathrm{Pol}(z)$ to $\mathrm{Pol}(z; k)$.

By setting $z = z_i$, we obtain with $\vec{b}_i \equiv \vec{R}(z_i)$, $(P(z_i) = 0)$,

$$\delta\vec{b}_i = \vec{\xi} + \vec{b}_i(\xi^\dagger \cdot b_i) \quad \leftrightarrow \quad \vec{b}_i \in \mathbb{C}P^{N-1} \text{ for each } i.$$

In the case that no vortices coincide; $z_i \neq z_j$, $(i \neq j)$,
$\{\vec{b}_i\}$ correspond to the moduli 'function' $\vec{R}(z)$ by one to one.

$$\vec{b}_i \quad \leftrightarrow \quad \vec{R}(z) = \sum_{i=1}^{k} \vec{b}_i \, e_{\rm p}^i(z).$$

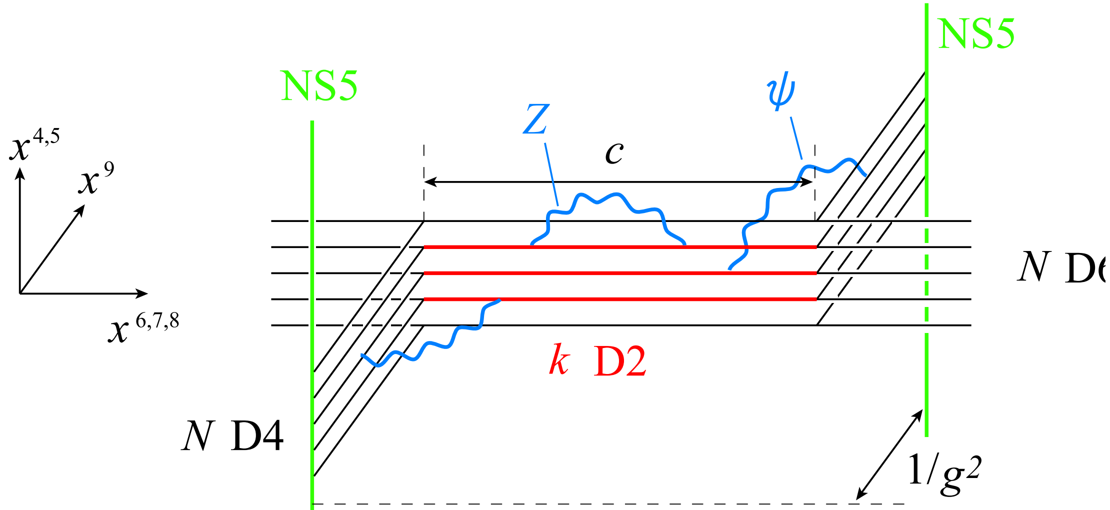
where, $e_{\rm p}^i(z)$ is a typical bases of Pol(z, k),

$$e_{\rm p}^i(z) \equiv \prod_{i=1, i\neq j}^{k} \left(\frac{z - z_j}{z_i - z_j} \right), \qquad \to \qquad e_{\rm p}^i(z_j) = \delta_j^i.$$

Furthermore, we find that $H_0|_{\rm multi} \simeq H_0|_{\rm single}$ with an orientation $\vec{b}_i$ in the vicinity of the i-th vortex $|z - z_i| \ll |z - z_j|$, $(j \neq i)$,

§. D-brane construction for k vortices

Hanany-Tong ('03)



$d = 4$ theory

2 NS5 : 012345

N D6 : 0123 678

N D4 : 0123 9

vortices

k D2 : 0 3 8

Moduli spaces for k vortices, $\mathcal{M}_{N,k}$ are described by the Higgs branch on an $U(k)$ gauge theory on k D2, which consists of chiral multiplets.

Bosonic parts:

$$\text{adjoint scalar } k \times k \text{ matrix } Z \ (k \times k),$$

$$N \text{ fundamental scalars } \Psi, (k \times N).$$

with a D flatness condition:

$$\Psi\Psi^\dagger - [Z,\, Z^\dagger] = \frac{2\pi}{g^2}$$

Thus, the moduli space of vortices is $U(k)$ Kähler quotient.
Its dimension is

$$\dim(\mathcal{M}_{N,k}) = \underbrace{2k^2}_{Z} + \underbrace{2kN}_{\Psi} - \underbrace{k^2}_{U(k)} - \underbrace{k^2}_{\text{constraint}} = 2kN$$

which coincides with result of the index theorem.

cf. Abelian case: E.Winberg,
non-Abelian case: Hanany-Tong

- Relation between $H_0(z)$ and Z, Ψ of Hanany-Tong

In the typical patch, we read

$$H_0(z) = \begin{pmatrix} 1_{N-1} & \vec{R}(z) \\ 0 & P(z) \end{pmatrix} \quad \rightarrow P(z),\ \Psi(z)^{\mathrm{T}} \equiv \begin{pmatrix} \vec{R}(z) \\ 1 \end{pmatrix},$$

then, we can derive $k \times k$ matrix Z and $k \times N$ matrix Ψ as,

$$z\, e^i(z) \equiv e^j(z)\, \mathrm{Z}_j{}^i \quad \mathrm{mod}\ P(z)$$
$$\Psi(z) = e(z) \cdot \Psi$$

with a basis of $\mathrm{Pol}(z;k)$, $e^i(z)$. For instance, in the bases $e^i(z) = e^i_{\mathrm{p}}(z)$,

$$Z = \mathrm{diag}(z_1,\ z_2,\ z_3, \cdots, z_k), \quad \Psi^{\mathrm{T}} = \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_k \\ 1 & \cdots & 1 \end{pmatrix}.$$

Transition of general basis of $\mathrm{Pol}(z;k)$ with $U \in GL(k, \mathbb{C})$

$$e'^i(z) = e^j(z)\, U_j{}^i,$$
$$\rightarrow Z' = U^{-1} Z U, \qquad \Psi' = U^{-1}\Psi, \quad \text{(Kähler quotient)}$$

We believe that

these matrices Z, Ψ are equivalent to those in Hanany-Tong.