# Quantization and High Energy Unitarity in Orbifold Theories

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A. Mück, L. N., A. Pilaftsis, R. Rückl, Phys. Rev. D 71 (2004) 066004;
L. N., A. Pilaftsis, work in progress

Durham, July 2005

# Outline

- Mass Eigenmode Basis and Quantization
- Ward and Slavnov-Taylor Identities
- Generalized Equivalence Theorem and High Energy Unitarity
- High Energy Unitarity Bounds from a Coupled Channel Analysis

Carena, Tait and Wagner [Acta Phys. Polon. B33 (2002) 2355];

Csaki, Grojean and Murayama [Phys. Rev. D69 (2004) 055006];

del Aguila, Perez-Victoria and Santiago [JHEP 02 (2003) 051];

Chivukula, Dicus and He [Phys. Lett. B525 (2002) 175]

# Scenario

Pure 5D Yang-Mills theory compactified on an $S^1/\mathbb{Z}_2$ orbifold.
Quantum corrections of bulk fields induce Brane Kinetic Terms (BKT).

$$\mathcal{L}_{5\mathrm{D}} = -\frac{1}{4}\left[1 + r_c\,\delta(y)\right] F^a_{MN} F^{a\,MN} + \mathcal{L}_{5\mathrm{D\,GF}} + \mathcal{L}_{5\mathrm{D\,FP}}$$

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$$\mathcal{L}_{5\mathrm{D\,GF}} = -\left[1 + r_c\delta(y)\right]\frac{1}{2\xi}\left(F[A^a_M]\right)^2$$

$$\mathcal{L}_{5\mathrm{D\,FP}} = \left[1 + r_c\delta(y)\right]\bar{c}^a\,\frac{\delta F[A^a_M]}{\delta\theta^b}\,c^b$$

$$F[A^a_M] = \partial^\mu A^a_\mu \,-\, \xi\,\partial_5 A^a_5$$

$$S = \int d^4x \int_{-\pi R}^{\pi R} dy\, \mathcal{L}_{5\mathrm{D}}$$

$$= \int d^4x \int_{-\pi R}^{\pi R} dy \big[1 + r_c \delta(y)\big] \; \Big[ \frac{1}{4} F^a_{MN} F^{a\;MN} - \frac{1}{2\xi} \big(F[A^a_M]\big)^2 + \bar{c}^a \frac{\delta F[A^a_M]}{\delta \theta^b} c^b \Big]$$

$$A^a_\mu(x,y) = \sum_{n=0}^{\infty} A^a_{(n)\mu}(x)\, f_n(y)$$

$$A^a_5(x,y) = \sum_{n=1}^{\infty} A^a_{(n)5}(x)\, g_n(y)$$

Mass eigenmode functions $f_n(y)$ and $g_n(y)$ uniquely specified by the following set of equations.

$$\int_{-\pi R}^{\pi R} dy\ \left[1 + r_c \delta(y)\right]\, f_n(y) f_m(y) = \delta_{n,m} \qquad f_n(y) = f_n(-y)$$

$$\int_{-\pi R}^{\pi R} dy\ \left[1 + r_c \delta(y)\right]\, g_n(y) g_m(y) = \delta_{n,m} \qquad g_n(y) = -g_n(-y)$$

$$\left[\partial_5^2 + m_n^2\right] f_n(y) = 0 \qquad \partial_5\, f_n(y) \;=\; -m_n g_n(y)$$

$$\left[\partial_5^2 + m_n^2\right] g_n(y) = 0 \qquad \partial_5\, g_n(y) \;=\; m_n f_n(y)$$

# Cancellation of mixing terms

$$\mathcal{L}_{5\mathrm{D}} \supset \left[1 + r_c \delta(y)\right] \left[(\partial_5 \partial^\mu A^a_\mu) A^a_5 + (\partial^\mu A^a_\mu)(\partial_5 A^a_5)\right]$$

The effective 4D theory is absent of mixing between scalar and vector Kaluza-Klein modes, if the basis functions fulfil the following conditions.

$$\begin{aligned} \partial_5 \, f_n(y) &= -m_n g_n(y) \\ \partial_5 \, g_n(y) &= m_n f_n(y) \end{aligned}$$

These equations ensure the absence of total derivatives, e.g.

$$\int_{-\pi R}^{\pi R} dy \, \left[1 + r_c \delta(y)\right] \partial_5 f_n(y) = 0$$

Following set of equations sufficient to specify mass eigenmode functions $f_n(y)$ and $g_n(y)$ uniquely.

$$\int_{-\pi R}^{\pi R} dy\ \left[1 + r_c\delta(y)\right]\ f_n(y) f_m(y) = \delta_{n,m} \qquad f_n(y) = f_n(-y)$$

$$\int_{-\pi R}^{\pi R} dy\ \left[1 + r_c\delta(y)\right]\ g_n(y) g_m(y) = \delta_{n,m} \qquad g_n(y) = -g_n(-y)$$

$$\left[\partial_5^2 + m_n^2\right] f_n(y) = 0 \qquad \partial_5\, f_n(y) \ = \ -m_n g_n(y)$$

$$\left[\partial_5^2 + m_n^2\right] g_n(y) = 0 \qquad \partial_5\, g_n(y) \ = \ m_n f_n(y)$$

# Mass Eigenmode Basis

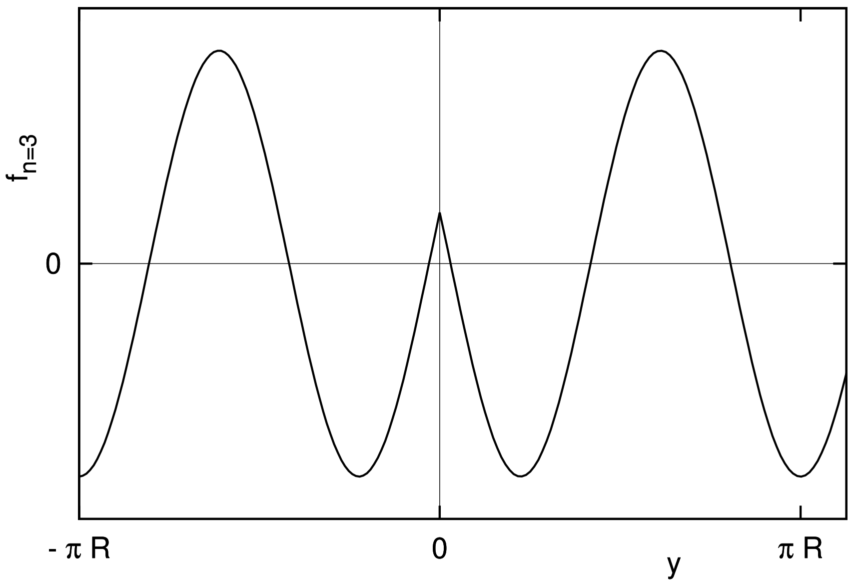
$$f_n(y) = \frac{N_n}{\sqrt{2^{\delta_{n,0}}\pi R}\ \cos m_n \pi R} \begin{cases} \cos m_n(y+\pi R) & \text{for } -\pi R < y \leq 0 \\ \cos m_n(y-\pi R) & \text{for } \quad 0 < y \leq \pi R \end{cases}$$

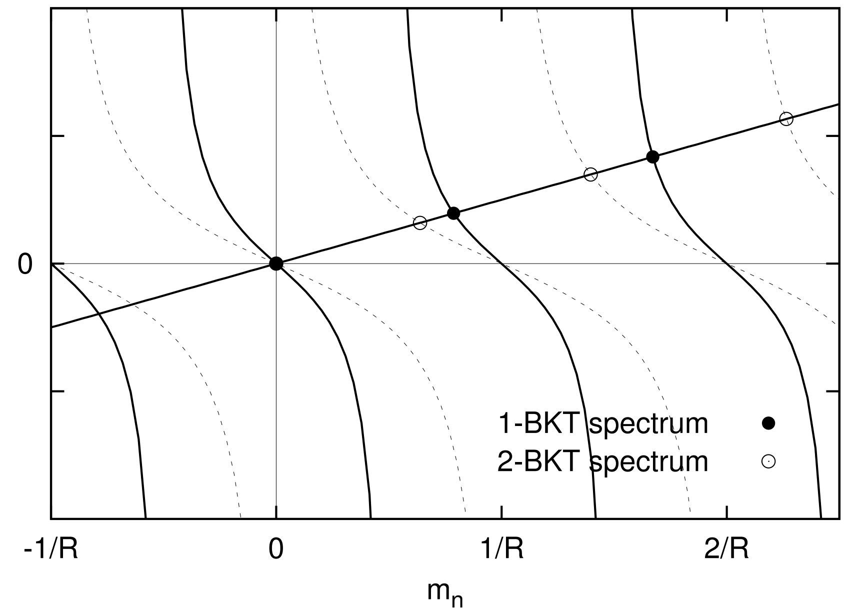
$$g_n(y) = \frac{N_n}{\sqrt{\pi R}\ \cos m_n \pi R} \begin{cases} \sin m_n(y+\pi R) & \text{for } -\pi R < y < 0 \\ \sin m_n(y-\pi R) & \text{for } \quad 0 < y \leq \pi R \\ 0 & \text{for } \quad y = 0 \end{cases}$$

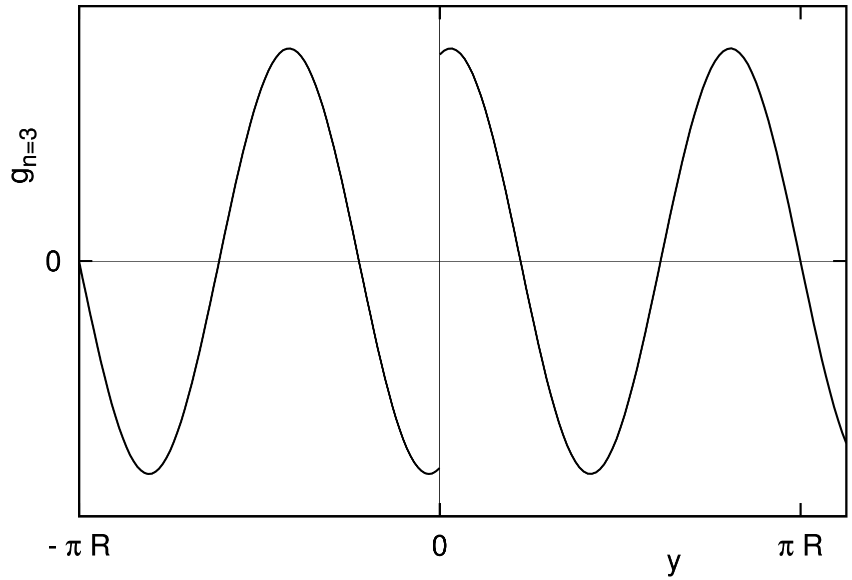
$$N_n^{-2} = 1 + \tilde{r}_c + \pi^2 R^2 \tilde{r}_c^2 m_n^2 \qquad \text{with} \quad \tilde{r}_c = \frac{r_c}{2\pi R} \geq 0$$

$$\frac{m_n r_c}{2} = -\tan m_n \pi R$$

# Mass Eigenmode Basis and Quantization




Mass spectrum of the effective theory.

Mass eigenmode functions $f_n(y)$ and $g_n(y)$ for $n = 3$.

No Regularization used in the derivation of the $f_n(y)$ and $g_n(y)$!

# Completeness of the Mass Eigenmode Basis

$$\delta(y_1 - y_2;\, r_c) = \sum_{n=0}^{\infty} \left[ f_n(y_1)\, f_n(y_2) + g_n(y_1)\, g_n(y_2) \right]$$

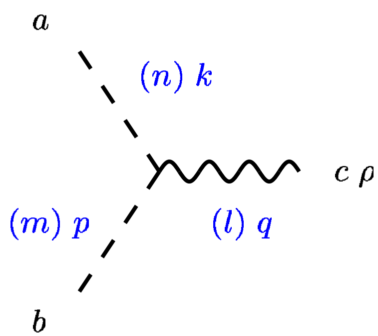
$$\int_{-\pi R}^{\pi R} dy\, [1 + r_c \delta(y)]\, h(y) \delta(y - y';\, r_c) = h(y')$$

$$\begin{aligned} \delta(y; r_c) &= \sum_{n=0}^{\infty} \frac{N_n^2}{2^{\delta_{n,0}} \pi R} \frac{\cos m_n (y \pm \pi R)}{\cos m_n \pi R} \\ &= \begin{cases} 1/r_c & \text{for} \quad y = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

# Fundamental interactions in the 4D effective theory

$$\Delta^{k,l,m,n} \equiv 4\pi R \int_{-\pi R}^{\pi R} dy\,[1 + r_c\delta(y)] g_k g_l g_m g_n$$

$$\Delta_n^{k,l} \equiv \sqrt{2^{\delta_{n,0}}\,4\pi R} \int_{-\pi R}^{\pi R} dy\,[1 + r_c\delta(y)] g_k g_l f_n$$




$$g\, f^{abc}\,\sqrt{2^{-1-\delta_{l,0}}}\,\Delta_l^{m,n}\,(p-k)_\rho$$

$$\lim_{r_c\to 0}\Delta_n^{k,l} = -\delta_{k+l+n,0} - \delta_{k+l-n,0} + \delta_{k-l+n,0} + \delta_{-k+l+n,0}$$

# Ward Identities

$$\delta A_M^a \;=\; \big(\,\delta^{ab}\partial_M \;-\; g_5 f^{abc} A_M^c\,\big)\,\theta^b\;, \qquad \Gamma[A_M^a] = \int d^4x \int_{-\pi R}^{\pi R} dy\; \mathcal{L}_{5\mathrm{D}}$$

Lagrangian $\mathcal{L}_{5\mathrm{D}}$ and hence tree-level effective action $\Gamma[A_M^a]$ invariant under 5D gauge transformations $\delta A_M^a$.

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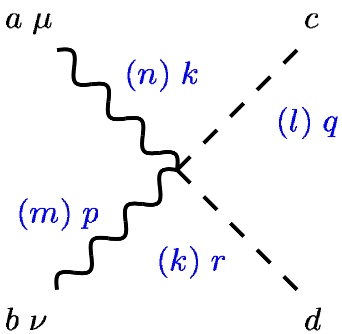
$$\partial_M \frac{\delta\Gamma}{\delta A^a_M} \;-\; g_5 f^{abc} \frac{\delta\Gamma}{\delta A^b_M} A^c_M = \; 0$$

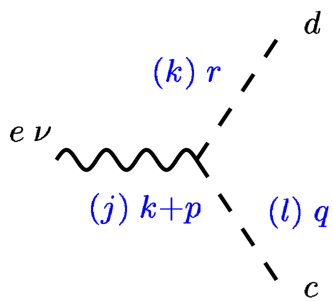
$$\partial_\mu \frac{\delta\Gamma}{\delta A^a_{(n)\mu}} \;+\; m_n \frac{\delta\Gamma}{\delta A^a_{(n)5}} \;=\; g f^{abc} \sum_{m,l=0}^{\infty} \sqrt{2}^{\;-1-\delta_{n,0}-\delta_{m,0}-\delta_{l,0}}$$
$$\times \left( \frac{\delta\Gamma}{\delta A^b_{(m)\mu}} A^c_{(l)\mu} \Delta_{m,n,l} \;+\; \frac{\delta\Gamma}{\delta A^b_{(m)5}} A^c_{(l)5} \Delta^{m,l}_{n} \right)$$

$-ik^{\mu}$

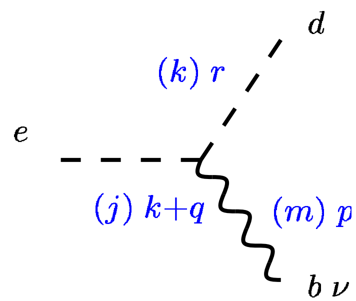



$$= \sum_{j=0}^{\infty} g\,\sqrt{2}^{\,-1-\delta_{n,0}-\delta_{j,0}}\Bigg[\sqrt{2}^{\,-\delta_{m,0}}\,\Delta_{m,n,j} f^{abe}$$

$\times$




$+\ \Delta_n^{l,j} f^{ace}$

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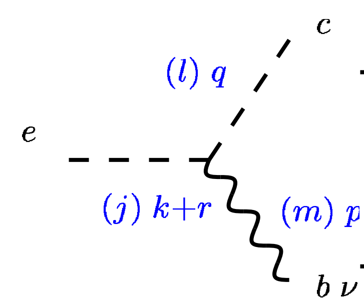


$\Bigg]$

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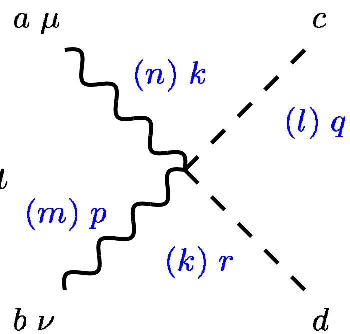


$$= \sum_{j=0}^{\infty} g \sqrt{2}^{\,-1-\delta_{n,0}-\delta_{j,0}} \Bigg[ \sqrt{2}^{\,-\delta_{m,0}} \, \Delta_{m,n,j} f^{abe}$$

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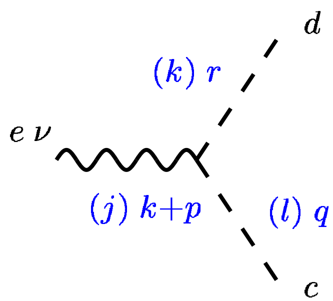


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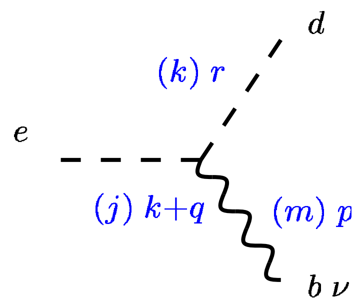


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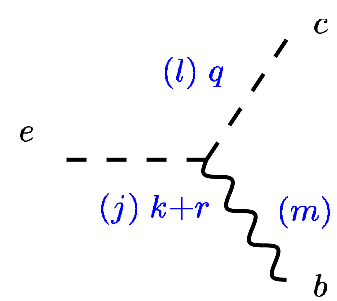


$$\Bigg]$$

$$\Delta_{l,n}^{k,m} = \sum_{j=0}^{\infty} 2^{-\delta_{j,0}} \Delta_{l}^{k,j} \Delta_{n}^{m,j}$$

# Proof of the Sum Rules

$$\sum_{j=0}^{\infty} 2^{-\delta_{j,0}} \Delta_l^{k,j} \Delta_n^{m,j} = \Delta_{l,n}^{k,m}$$

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$$\sum_{j=0}^{\infty} \Big( \int_{-\pi R}^{\pi R} dy\, [1 + r_c \delta(y)] f_l(y) g_k(y) g_j(y) \Big)$$

$$\times \Big( \int_{-\pi R}^{\pi R} dy'\, [1 + r_c \delta(y')] f_n(y') g_m(y') g_j(y') \Big) =$$

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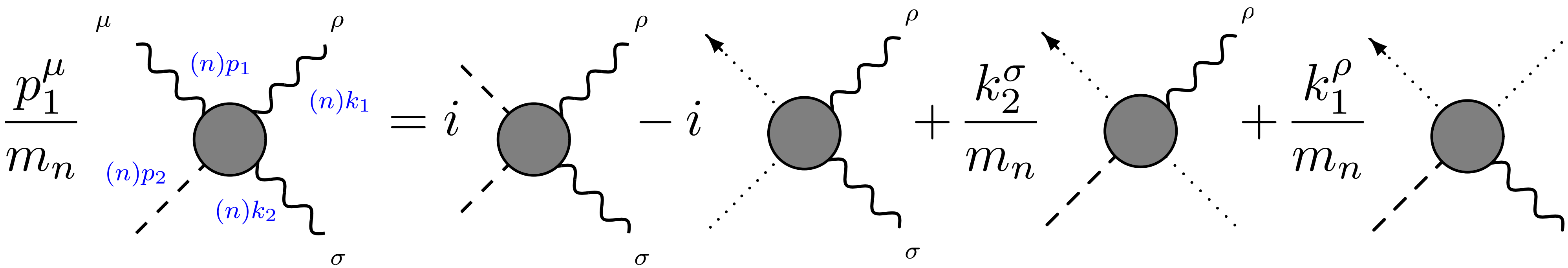
$$\int_{-\pi R}^{\pi R} dy\, [1 + r_c \delta(y)] f_l(y) f_n(y) g_k(y) g_m(y)$$

$$\sum_{j=0}^{\infty} \left[ g_j(y)\, g_j(y') + f_j(y)\, f_j(y') \right] = \delta(y - y';\, r_c)$$

# Slavnov-Taylor Identities

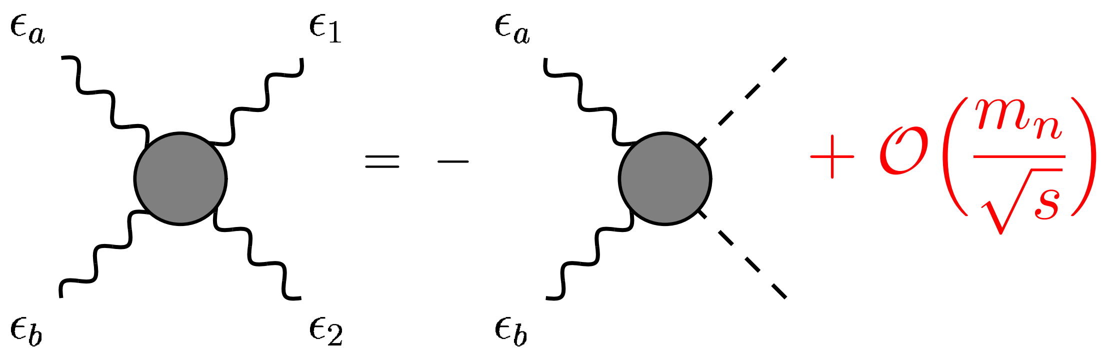
- gauge invariance of classical 5D Lagrangian
  $\Rightarrow$ Ward Identities for fundamental interactions

- BRS invariance of gauge-fixed 5D Lagrangian
  $\Rightarrow$ Slavnov-Taylor Identities for 1PI Green's functions, e.g.



# Equivalence Theorem

for elastic scattering process $A^a_{(n)T} A^b_{(n)T} \to A^c_{(n)L} A^d_{(n)L}$



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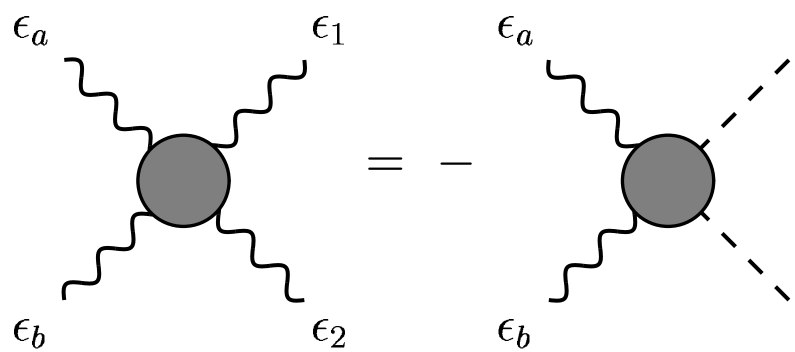
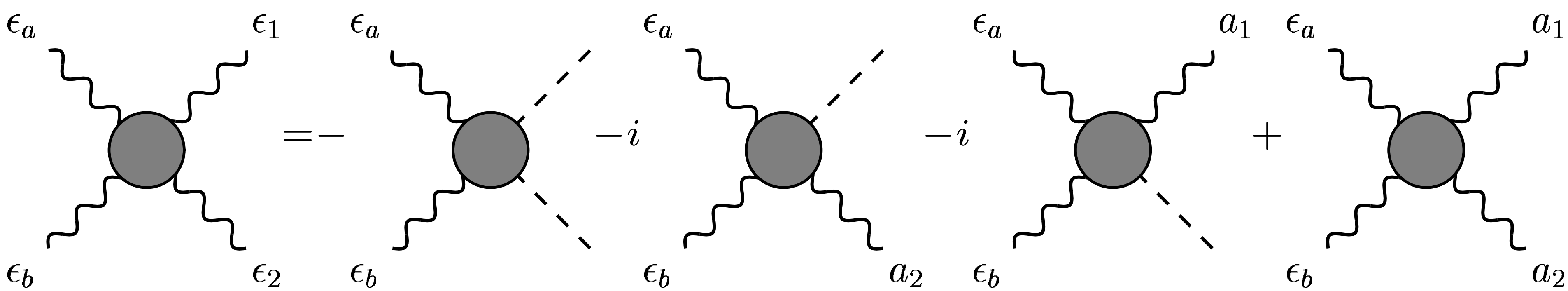



$$+ \mathcal{O}\left(\frac{m_n}{\sqrt{s}}\right)$$

compactified orbifold theories: **scalar KK modes** $A^a_{(n)5}$ $\longleftrightarrow$ spont. broken gauge theories: **would-be Goldstone bosons**

# Generalized Equivalence Theorem

for elastic scattering process $A^a_{(n)T} A^b_{(n)T} \longrightarrow A^c_{(n)L} A^d_{(n)L}$




form of the longitudinal polarization vectors:

$$\epsilon^\mu_{1,2} = \frac{1}{m_n} k^\mu_{1,2} + a^\mu_{1,2} \qquad\qquad a^\mu_{1,2} \sim \mathcal{O}\left(\frac{m_n}{\sqrt{s}}\right)$$

# Generalized Equivalence Theorem

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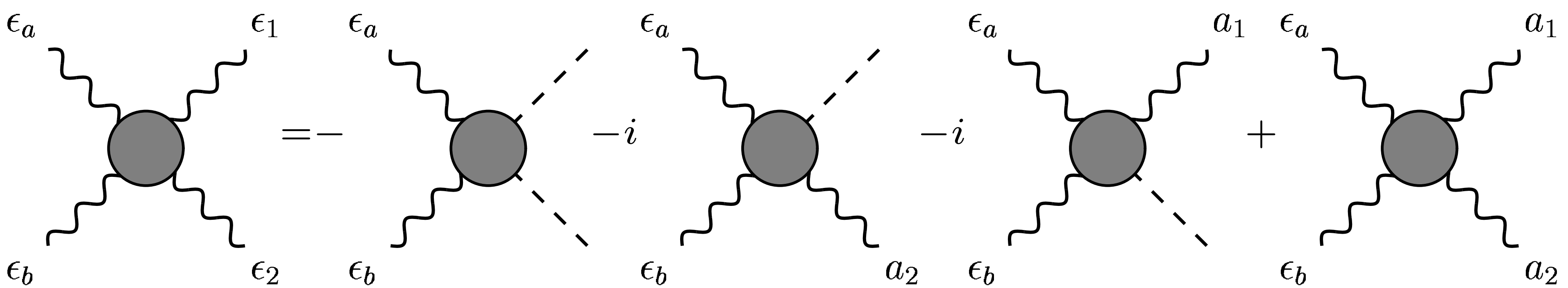
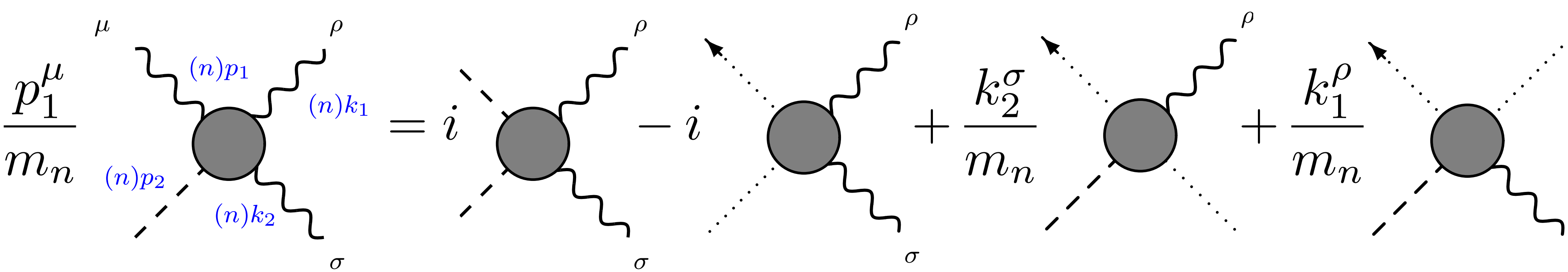



**Proof** based on Slavnov-Taylor identities, such as



# High Energy Unitarity

- bounds on **total cross-section**

  **Froissart, Martin** [PR 123 (1961) 1053; PR 129 (1963) 1432]:
  In the high-energy limit, total cross-sections can not rise faster than a logarithm of the energy.

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- bounds on **partial wave amplitudes**

  **Lee, Quigg, Thacker** [PRD 16 (1977) 1519]:
  Used to find an upper bound on the energy, up to which the 4D effective theory makes correct predictions.

# Coupled Channel Analysis

$$S^\dagger S = \mathbf{1}$$

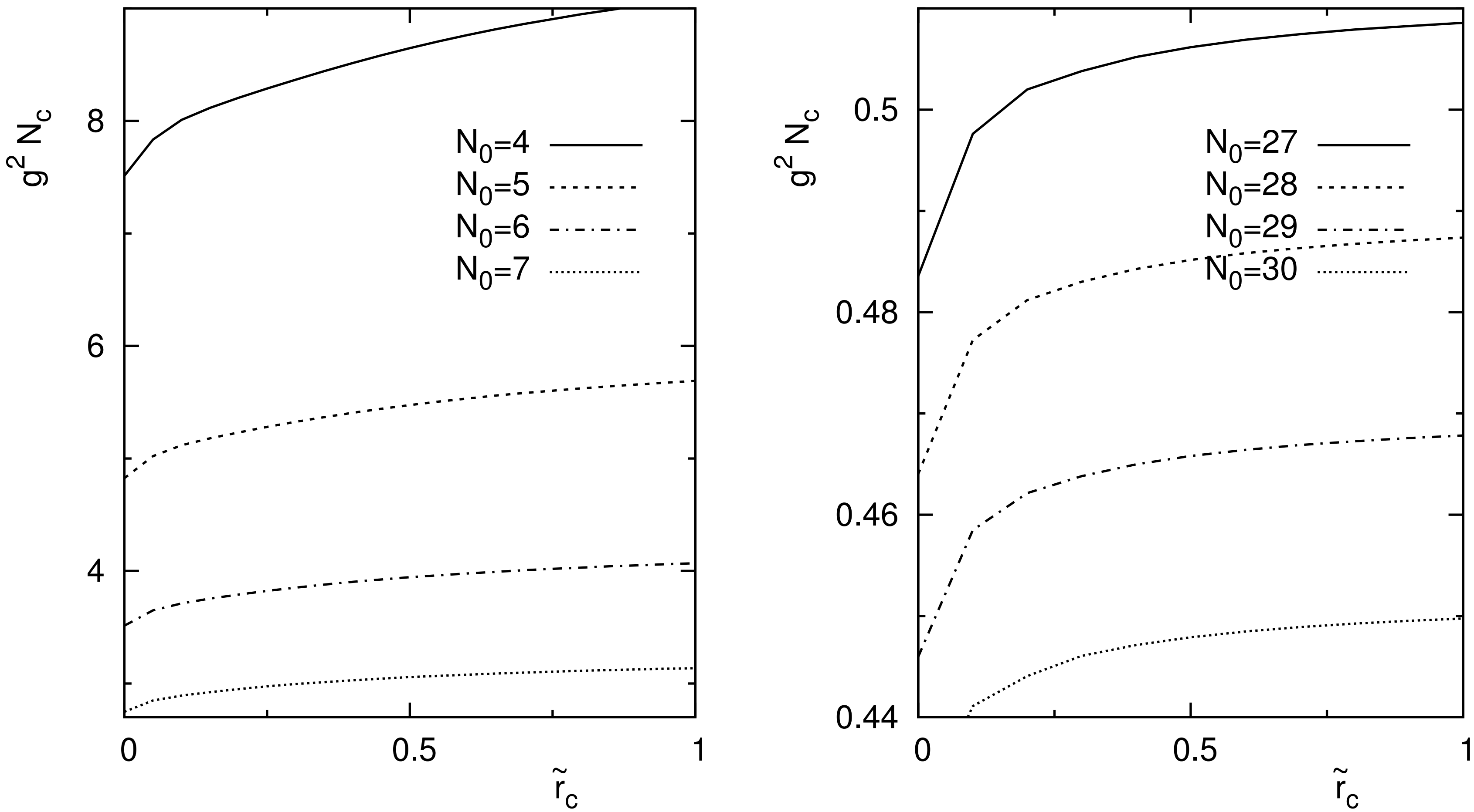
$$2\,\mathrm{Im}\,T_{fi} = \sum_j T_{fj} T^*_{ji}$$ optical theorem

$$\mathrm{Im}\,[a_0]_{fi} = \sum_j \sigma_j [a_0]_{fj} [a_0]^*_{ji}$$ $$a_0 \equiv \frac{1}{32\pi} \int_{-1}^{1} d\cos\theta\; T$$

$$\mathrm{Im}\,\tilde{a}_0 = \tilde{a}_0 \tilde{a}_0^*$$ $$[\tilde{a}_0]_{ij} \equiv \sqrt{\sigma_i \sigma_j}\,[a_0]_{ij}$$

$$\mathrm{Im}\,\alpha = \alpha\alpha^*$$ $\alpha \equiv$ max. eigenvalue of $\tilde{a}_0$

$$|\alpha| \leq 1$$ $$\mathrm{Im}\,\alpha \leq |\alpha|$$




High energy unitarity bound on the KK modes.

# Conclusions

- Orbifold theories with Brane Kinetic Terms (BKT) respect
  High Energy Unitarity

  Quantization $\longrightarrow$ Ward and Slavnov-Taylor identities
  $\longrightarrow$ Generalized Equivalence Theorem $\longrightarrow$ High Energy Unitarity

- Unitarity Bound not sensitive to size of BKT,
  but BKT crucial for decays of KK modes.

- Flexibility of the approach:
  can deal with more complex BKT structures, 6D orbifold theories

## Flexibility of the approach

The approach can easily be modified for more complex BKT structures, 6D orbifold theories or warped extra dimensions.
For example, two BKT at $y = 0$ and $y = \pi R$ :

$$\int_{-\pi R}^{\pi R} dy \left[1 + r_c \delta(y) + r_c \delta(y - \pi R)\right] f_n(y) f_m(y) = \delta_{n,m}$$

$$\int_{-\pi R}^{\pi R} dy \left[1 + r_c \delta(y) + r_c \delta(y - \pi R)\right] g_n(y) g_m(y) = \delta_{n,m}$$

$$f_n(y) = f_n(-y) \qquad \left[\partial_5^2 + m_n^2\right] f_n(y) = 0 \qquad \partial_5\, f_n(y) \;=\; -m_n g_n(y)$$

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